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NONLINEAR SCHRÖDINGER EQUATION IN A HYDROELASTICITY PROBLEM

A. N. Volobuev and A. P. Tolstonogov

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A system of hydrodynamic equations is solved for an inviscid liquid flowing in an elastic pipe. It is shown that this system is equivalent to a nonlinear Schrödinger equation. The solution is considered as applied to the development of hydroelastic components of hydraulic systems.

Calculation of the flow in an elastic pipeline is a problem of hydroelasticity. Some questions connected with this problem were treated, for example, in [1], where steady-state viscous flow in an elastic pipe was investigated within the framework of the linear equation of momentum. In that study a linear relation of the pipe radius to the pressure on the pipe walls was used.

In the design of hydraulic systems containing an elastic thin-walled pipeline, it is necessary to solve the problem of the propagation of a solitary wave (soliton) induced by ejection of a volume of liquid. Determination of the soliton shape is important for timely operation of the hydrorelay.

In [2] a method was proposed for determination of the soliton shape and it was shown that the problem could be reduced to solution of the KdV equation using Hooke's law written as $p = c\Delta S/S_0$.

However, practical calculations have shown that if rather large volumes are ejected, this law should be used in the slightly different form

$$p = -c\frac{\Delta S}{S}, \tag{1}$$

where $\Delta S = S - S_0$ and the minus sign indicates that with elastic deformations the force exerted on the liquid is directed opposite its motion along the pipeline.

Let us consider a solution of the hydrodynamic equations for an inviscid liquid flow in the form suggested in [3]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial (pS)}{S\partial x} = 0, \tag{2}$$

$$\frac{\partial S}{\partial t} + \frac{\partial (uS)}{\partial x} = 0. \tag{3}$$

Here, unlike [3], we retain the nonlinear convective term in the momentum equation.

The solution of momentum equation (2) and continuity equation (3) will be sought using the complex velocity potential $\varphi = \varphi(x, t)$. It will be expanded in a series by using the perturbation λ similarly to what is done in quantum mechanics in going from the Schödinger to the Hamilton-Jacobi equation [4]:

$$\varphi = \varphi_0 + \frac{\lambda}{i}\varphi_1 + \left(\frac{\lambda}{i}\right)^2\varphi_2 + \dots \tag{4}$$

Now define the function Ψ by the formula

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$$\Psi = \exp\left(\frac{i}{\lambda}\varphi\right). \tag{5}$$

The use of the first two terms in series (4), gives

$$\Psi = |\Psi| \exp\left(\frac{i}{\lambda}\varphi_0\right). \tag{6}$$

Here $|\Psi| = \exp(\varphi_1)$ is the modulus of the function Ψ, $u = \partial\varphi_0/\partial x$ is the liquid velocity in the pipeline, since φ_0 is the real part of the velocity potential up to λ^2.

If $|\Psi| = (S/S_0)^{1/2}$ is assumed, then with the use of (1) the equality $pS = -c(S - S_0) = -cS_0(|\Psi|^2 - 1)$ can be written. The last term in Eq. (2) takes the form:

$$\frac{1}{\rho}\frac{\partial (pS)}{S\partial x} = -2a^2\frac{\partial}{\partial x}(\ln |\Psi|) = -2a^2\frac{\partial\varphi_1}{\partial x}, \tag{7}$$

where $a = \sqrt{c/\rho}$ is the velocity of propagation of a pressure wave along the pipeline and $\varphi_1 = \ln |\Psi|$.

With the use of the velocity potential, momentum equation (2) can be integrated once; then the system of equations (2) and (3) takes the form

$$\frac{\partial\varphi_0}{\partial t} + \frac{u^2}{2} - 2a^2\varphi_1 = 0, \tag{8}$$

$$\frac{\partial |\Psi|^2}{\partial t} + \frac{\partial u |\Psi|^2}{\partial x} = 0. \tag{9}$$

In the integration the right-hand side of Eq. (8) is assumed to be zero due to the appropriate choice of the initial level of the potential φ_0 [5].

Next, it will be shown that system (8) and (9) is equivalent to the Schrödinger nonlinear equation. To do this, we will consider the unsteady-state Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} + \frac{\lambda}{2}\frac{\partial^2\Psi}{\partial x^2} = -\frac{2a^2}{\lambda}\varphi_1\Psi. \tag{10}$$

Bearing in mind that

$$\frac{\partial\Psi}{\partial t} = \frac{i}{\lambda}\Psi\frac{\partial\varphi}{\partial t} \quad \text{and} \quad \frac{\partial^2\Psi}{\partial x^2} = -\frac{1}{\lambda^2}\Psi\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{i}{\lambda}\Psi\frac{\partial^2\varphi}{\partial x^2},$$

proceeding from (10), we obtain after separation of the real and imaginary parts:

$$\frac{\partial\varphi_0}{\partial t} + \frac{1}{2}\left(\frac{\partial\varphi_0}{\partial x}\right)^2 - 2a^2\varphi_1 = \frac{\lambda^2}{2}\left[\left(\frac{\partial\varphi_1}{\partial x}\right)^2 + \frac{\partial^2\varphi_1}{\partial x^2}\right] = 0\,(\lambda^2), \tag{11}$$

$$\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_1}{\partial x} + \frac{1}{2}\frac{\partial^2\varphi_0}{\partial x^2} = 0. \tag{12}$$

Equation (12) is completely equivalent to (9), if it is taken into account that

$$\frac{\partial \varphi_1}{\partial t} = \frac{1}{2\,|\Psi|^2} \frac{\partial\,|\Psi|^2}{\partial t} \quad \text{and} \quad \frac{\partial \varphi_1}{\partial x} = \frac{1}{2\,|\Psi|^2} \frac{\partial\,|\Psi|^2}{\partial x}.$$

Just as in series (4), only terms linear in the perturbation λ were retained in Eq. (11). Thus, Eqs. (11), (12) and, consequently, (10) are completely equivalent to system (8) and (9).

The nonlinear Schrödinger equation is written in the form

$$i\frac{\partial \Psi}{\partial t} = \frac{a}{k} \frac{\partial^2 \Psi}{\partial x^2} = -\,\omega\,(\ln\,|\Psi|)\,\Psi\,, \tag{13}$$

where $\omega = 2a^2/\lambda$ is the cyclic frequency; $k = 2a/\lambda$ is the wave number.

Following [6], the solution of Eq. (13) will be sought as

$$\Psi = f\,(kx - \omega t)\,\exp\,[i\,(rx - \delta t)]\,, \tag{14}$$

where the constants r and δ as well as the function $f(kx - \omega t)$ are unknown so far. With (6) taken into account, it can be concluded that $|\Psi| = f(kx - \omega t)$.

Substitution of (14) into (13) gives

$$akf'' + if'\,(-\,\omega + 2ar) + f\left(\delta - \frac{ar^2}{k}\right) + \omega f \ln f = 0\,. \tag{15}$$

In Eq. (15) differentiation is performed with respect to the variable $\xi = kx - \omega t$ and must not have imaginary terms since the function $f = |\Psi|$ is a real number. Assuming $r = \omega/2a$ and bearing in mind that $\omega = ak$, we have

$$f'' + f\left(\frac{\delta}{\omega} - \frac{1}{4}\right) + f \ln f = 0\,. \tag{16}$$

The solution of Eq. (16) is sought in the form

$$f = C_1 \exp\,[C_2\,(kx - \omega t)^2/2]\,. \tag{17}$$

The substitution of (17) into (16) gives

$$C_2 + \left(\frac{\delta}{\omega} - \frac{1}{4}\right) + \ln C_1 + (kx - \omega t)^2 \left(C_2^2 + \frac{C_1}{2}\right) = 0\,. \tag{18}$$

The last term in Eq. (18) should not depend on the coordinate x the time t, i.e., $C_2 = 1/2$; then $C_1 = \exp\,(3/4 - \delta/\omega)$.

Consequently,

$$f = |\Psi| = \exp\left(\frac{3}{4} - \frac{\delta}{\omega}\right) \exp\,[-\,(kx - \omega t)^2/4]\,. \tag{19}$$

In view of the fact that the cross-sectional area is $S = S_0|\Psi|^2$ and in view of the boundary condition $S = S_0$ at $x \to \pm\infty$, we finally find the shape of the solitary wave propagating along the elastic pipeline with ejection of large volumes of liquid:

$$S = S_0 + S_A \exp\,[-\,(kx - \omega t)^2/2]\,,$$

where $S_A = \Delta S_{max}$ corresponds to the maximum value of the additional area in (1):

$$S_A = S_0 \exp\left(\frac{3}{2} - \frac{2\delta}{\omega}\right).$$

Here δ depends on the characteristic of the pipeline material. At large δ, typical of rigid pipes, S_A tends to zero. Small values of δ are typical of an elastic pipeline.

The value of the excess pressure in the pipeline is determined using Eq. (1):

$$p = -c\frac{S - S_0}{S} = -c\left\{1 - 1\Big/\left(1 + \frac{S_A}{S_0}\exp\left[-(kx - \omega t)^2/2\right]\right)\right\}.$$

For large increments of the area ΔS we have

$$p = -c\frac{S_A}{S}\exp\left[-(kx - \omega t)^2/2\right] = p_{max}\exp\left[-(kx - \omega t)^2/2\right], \tag{20}$$

where p_{max} is the maximum excess pressure in the soliton. The shape of the soliton and the pressure in it follow a Gaussian curve.

It follows from (20) that the momentum of the pressure and the cross-sectional area appear similar to each other, just as in the case where the KdV model was used [2]. However, unlike the solitary waves described by the KdV equation, in this case, where Hooke's law was used in the form of (1), a nondispersed wave, propagating with the velocity $a = \sqrt{c/\rho}$, is obtained. This allows a simpler design of hydraulic relays in hydroelastic components of hydraulic systems.

NOTATION

C_1 and C_2, constants; c, characteristic of the material of the pipe (elasticity); i, imaginary unit; p, excess pressure on the pipe wall; S, instantaneous cross-sectional area of the pipe; S_0, undisturbed cross-section of the pipe; ΔS, increment of the cross-sectional area of the pipe; ρ, liquid density.

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